Hom-Tensor Adjunction

Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be functors. We say that L **is left adjoint to** R, R **is right adjoint to** L, or L **and** R **are adjoint** if there is a natural isomorphism

$$\tau : \mathrm{Hom}_{\mathcal{B}}(L(-), -) \to \mathrm{Hom}_{\mathcal{A}}(-, R(-)).$$

That is, given $\varphi : A \to A'$ in $\mathcal{A}$ and $\psi : B \to B'$ in $\mathcal{B}$, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) & \xrightarrow{L\varphi^*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B) & \xrightarrow{\psi_*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \\
\downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) & \xrightarrow[\varphi^*]{} & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) & \xrightarrow[R\psi_*]{} & \mathrm{Hom}_{\mathcal{A}}(A, R(B')).
\end{array}$$

Exercise 1 Fix rings R and S and fix an $R-S$ bimodule B. Show that the functors $- \otimes_R B$ and $\mathrm{Hom}_S(B, -)$ are adjoint.

Let A be a right R-module and let C be a right S-module. We need to show that

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

naturally.

Let $f \in \mathrm{Hom}_S(A \otimes_R B, C)$; i.e., $f : A \otimes_R B \to C$. Define

$$\tau : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

by declaring τf to be the map that at $a \in A$ outputs the map $b \mapsto f(a \otimes b) \in C$; i.e.,

$$(\tau f(a))(b) = f(a \otimes b).$$

Let $g \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$; i.e., $g : A \to \mathrm{Hom}_S(B, C)$. Define

$$\sigma : \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \to \mathrm{Hom}_S(A \otimes_R B, C)$$

by declaring σg to be the map that is induced by the bilinear map $A \times B \to C$, $(a, b) \mapsto (g(a))(b)$; i.e.,

$$(\sigma g)(a \otimes b) = (g(a))(b).$$

We claim that σ and τ are inverses. To see this, observe that

$$(\sigma\tau f)(a \otimes b) = (\tau f(a))(b) = f(a \otimes b)$$

and

$$(\tau\sigma g(a))(b) = \sigma g(a \otimes b) = (g(a))(b),$$

and thus σ and τ are inverses, as we claimed.

$$\begin{array}{ccccc} \mathrm{Hom}_S(A' \otimes_R B, C) & \xrightarrow{(\varphi\otimes B)^*} & \mathrm{Hom}_S(A \otimes_R B, C) & \xrightarrow{\psi_*} & \mathrm{Hom}_S(A \otimes_R B, C') \\ \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\ \mathrm{Hom}_R(A', \mathrm{Hom}_S(B, C)) & \xrightarrow[\varphi^*]{} & \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) & \xrightarrow[(\psi_*)_*]{} & \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C')). \end{array}$$

Next, we claim this isomorphism is natural. We begin by showing the commutivity of the left square. We must show that given an S-module homomorphism $\varphi : A \to A'$ and an R-module homomorphism $\psi : C \to C'$, if $f \in \mathrm{Hom}_S(A' \otimes_R B, C)$, then

$$\tau(\varphi \otimes B)^*(f) = \varphi^*\tau(f)$$

in $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$. Observe as

$$\Big(\big(\tau(\varphi \otimes B)^* f\big)(a)\Big)(b) = \big((\varphi \otimes B)^* f\big)(a \otimes b) = \big(f(\varphi \otimes B)\big)(a \otimes b) = f\big(\varphi(a) \otimes b\big),$$

while

$$\Big(\big(\varphi^*\tau f\big)(a)\Big)(b) = \Big(\tau f\big(\varphi(a)\big)\Big)(b) = f\big(\varphi(a) \otimes b\big).$$

Thus the left square commutes.

To see the commutivity of the right square, we must show that if $f \in \mathrm{Hom}_S(A \otimes_R B, C)$, then

$$\tau\psi_*(f) = (\psi_*)_*\tau(f)$$

in $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C'))$. Observe as

$$\big(\tau\psi_* f(a)\big)(b) = \psi_* f(a \otimes b) = \psi f(a \otimes b),$$

while

$$\big((\psi_*)_*\tau f(a)\big)(b) = \big(\psi_*\tau f(a)\big)(b) = \big(\psi\tau f(a)\big)(b) = \psi f(a \otimes b).$$

Thus the right square commutes.